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Normal Random Processes

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Georg Lindgren

Papers summarized in the dissertation

- [A] Lindgren, G.: Some properties of a normal process near a local maximum. Ann. Math. Statist. 41, 1870-1883 (1970)
- [B] - - - - - : Extreme values of stationary normal processes. Z. Wahrscheinlichkeitstheorie verw. Geb. 17, 39-47 (1971)
- [C] - - - - - : Wave-length and amplitude in Gaussian noise. To appear in Advances in Applied Probability (1972)
- [D] - - - - - : Wave-length and amplitude for a stationary Gaussian process after a high maximum. To appear in Z. Wahrscheinlichkeitstheorie verw. Geb.
- [E] - - - - - : Local maxima of Gaussian fields. To appear in Arkiv för matematik

1. INTRODUCTION

In the theory of stationary stochastic processes with continuous time, much effort has been devoted to the problem of finding the statistical distribution of wave-characteristics such as the time between crossings of a level, and the distance between a local maximum and the following minimum, quantities which are important for the physical appearance of the process paths.

Our main concern will be the wave-length and amplitude after a local maximum of a Gaussian process $\xi(t)$, when the maximum has a prescribed height u . The terms wave-length and amplitude will mean the horizontal and the vertical distances from the maximum to the next minimum.

Of the papers summarized here, [A], [B], [C], and [D] deal with the wave-problem for a random process developing in time, i.e. the process $\xi(t)$ has a one-dimensional parameter t . Paper [E] generalizes the results and methods to the more complicated situation with a random surface, i.e. to the case when t is n -dimensional.

The literature on crossing problems is wide and scattered. No complete review will be given here, only some indications of the connection between the present papers and the earlier works. The main reference for the analytical aspects is the book by Cramér and Leadbetter, *Stationary and related stochastic processes*, Wiley, New York (1967).

Of the problems treated in this summary, the wave-length has the deepest roots in the literature. In his now classical work, *Mathematical analysis of random noise*, Bell System Tech. J. 24, 46-156 (1945), S. O. Rice approximated the distribution of the time between successive crossings of the zero level. His method is essentially equivalent to the moment-bound method which has been frequently used to approximate the distribution of the distance between events in a point process. See, e.g., Longuet-Higgins, *The distribution of intervals between zeros of a stationary random function*, Phil. Trans. Royal Soc. London, Ser. A 254, 557-599 (1962), which contains many interesting illustrations, and also Cramér, Leadbetter, and Serfling, *On distribution function - moment relationships in a stationary point process*, Z. Wahrscheinlichkeitstheorie verw. Geb. 18, 1-8 (1971), which gives a modern, condensed, mathematical summary. In [C] we have used a

similar technique to approximate the wave-length distribution.

The main tool in our papers is a conditioned process $\xi_u(t)$, which plays the role of $\xi(t)$ given that a maximum with height u has occurred at 0. The representation of $\xi_u(t)$ as a sum of two different stochastic processes was inspired by a similar result by Slepian for processes conditioned by the presence of zero-crossings; see, *On the zeros of Gaussian noise*, Time series analysis (M. Rosenblatt ed.), Wiley, New York (1962).

Despite its great practical importance, the amplitude is discussed very little in the literature, and practically nothing seems to be in print as far as its distribution is concerned.

2. TOOLS: THE CONDITIONED PROCESS $\xi_u(t)$

Let $\{\xi(t), t \in \mathbb{R}\}$ be a separable, stationary Gaussian process with mean zero, unit variance, and covariance function

$$r(t) = \text{Cov}(\xi(s+t), \xi(s)).$$

The infinitesimal structure of the sample paths of ξ is closely connected with the behaviour of $r(t)$ as $t \rightarrow 0$. Suppose r is four times continuously differentiable with $\lambda_2 = -r''(0)$, $\lambda_4 = r^{IV}(0)$, and that, furthermore,

$$r^{IV}(t) = \lambda_4 - O(|\log|t||^{-a}) \quad \text{as } t \rightarrow 0 \quad \text{for some } a > 1. \quad (1)$$

Then ξ has, with probability one, twice continuously differentiable sample paths.

When discussing the behaviour of $\xi(t)$ near a local maximum with height u , one has to be careful. Since the event " ξ has a local maximum at t_0 with height u " has probability zero, there is no immediate interpretation of conditional distributions, given that this event has occurred.

In [A], the ergodic definition is stated, i.e. the definition which can be interpreted by the aid of relative frequencies in a single, very long, realization of ξ . To summarize, let η_u be a random variable (r.v.) with the density

$$q_u^*(y) = k_u \{y + \lambda_2 u / (\lambda_4 - \lambda_2^2)\} \exp\{-(\lambda_4 - \lambda_2^2)y^2/2\},$$

$$y \geq -\lambda_2 u / (\lambda_4 - \lambda_2^2),$$

and let $\{\Delta(t), t \in \mathbb{R}\}$ be a separable, non-stationary, zero-mean Gaussian process, independent of η_u , and with the covariance function

$$C(s, t) = r(s-t) - \{\lambda_2(\lambda_4 - \lambda_2^2)\}^{-1} \left\{ \lambda_2 \lambda_4 r(s)r(t) + \lambda_2^2 r(s)r''(t) \right. \\ \left. + (\lambda_4 - \lambda_2^2)r'(s)r'(t) + \lambda_2^2 r''(s)r(t) + \lambda_2 r''(s)r''(t) \right\}.$$

If r satisfies (1), then Δ has, with probability one, twice continuously differentiable sample paths. Finally define the process

$$\xi_u(t) = ur(t) - \eta_u(\lambda_2 r(t) + r''(t)) + \Delta(t). \quad (2)$$

One major point in [A] is the conclusion that $\xi_u(t)$ can serve as a model for the stochastic behaviour of ξ near a local maximum with height u . Theorems 1 and 2 state:

HORIZONTAL WINDOW PROPERTY: If the event $A(h, h')$ is defined by

$$A(h, h'): \xi' \text{ has a down-crossing zero at some time } s \in (-h', 0) \text{ and } \xi(s) \in (u, u+h),$$

then

$$\begin{aligned} P(\xi_u(t_i) \leq x_i, i = 1, \dots, n) &= \\ &= \lim_{h \rightarrow 0} \lim_{h' \rightarrow 0} P(\xi(t_i) \leq x_i, i = 1, \dots, n \mid A(h, h')). \end{aligned}$$

ERGODIC PROPERTY: If the process ξ is ergodic, and

$$N_T(h) = \text{the number of local maxima } s \text{ of } \xi \text{ in } (0, T] \text{ such that } \xi(s) \in (u, u+h)$$

$$N_T(x; h) = \text{the number of local maxima } s \text{ of } \xi \text{ in } (0, T] \text{ such that } \xi(s) \in (u, u+h) \text{ and } \xi(s+t_i) \leq x_i, i = 1, \dots, n$$

then, with probability one,

$$P(\xi_u(t_i) \leq x_i, i = 1, \dots, n) = \lim_{h \rightarrow 0} \lim_{T \rightarrow \infty} \frac{N_T(x; h)}{N_T(h)} = \lim_{h \rightarrow 0} \frac{E(N_1(x; h))}{E(N_1(h))}.$$

The wave-characteristics wave-length and amplitude for ξ_u are defined by

$$\tau_u = \text{the first local minimum of } \xi_u$$

$$\delta_u = u - \xi_u(\tau_u),$$

and their distributions are the same as the ergodic distributions of the wave-length and amplitude after local u -maxima of ξ .

Paper [D] deals with the asymptotic behaviour of ξ_u as $u \rightarrow +\infty$, i.e. when the maximum of ξ is very high. Then the dominant term in $\xi_u(t)$ is $ur(t)$, while the stochastic terms $-\eta_u(\lambda_2 r(t) + r''(t))$ and $\Delta(t)$ only serve as stochastic perturbations of small or moderate order. It holds:

The r.v. η_u is asymptotically normal with mean zero and variance $(\lambda_4 - \lambda_2^2)^{-1}$ as $u \rightarrow \infty$.

When looking for the first local minimum of ξ_u , we have to distinguish between different types of covariance functions:

- a) r has a local minimum at some time $t_0 > 0$
- b) r has no local minimum in $(0, \infty)$.

Case a) can be further subdivided according to whether r has only one, or more than one local minimum; in the latter case we let t_0 be the smallest one. Furthermore, $r'(t)$ can be zero without $r(t)$ having a local minimum. Theorem 4.1 in [D] treats this case (which is of mere theoretical interest).

Case a) and its different subcases are dealt with in [D], Sections 3 and 4, while case b) is treated in [D], Section 5-7. We sum up some of the results.

a) Suppose r has a local minimum at $t_0 > 0$ with a first non-vanishing derivative $r^{(2k_0)}(t_0) > 0$, and that $r'(t) < 0$ for $0 < t < t_0$. Then $P(|\tau_u - t_0| > \varepsilon) \rightarrow 0$ as $u \rightarrow \infty$, i.e. $\tau_u \xrightarrow{P} t_0$. We can therefore expand $\xi_u(\tau_u)$ and $\xi_u'(\tau_u)$ in Taylor series, e.g.

$$\begin{aligned} 0 = \xi_u'(\tau_u) &= ur'(\tau_u) - \eta_u(\lambda_2 r'(\tau_u) + r'''(\tau_u)) + \Delta'(\tau_u) = \\ &= \frac{u(\tau_u - t_0)^{2k_0-1}}{(2k_0-1)!} r^{(2k_0)}(t_0) - \eta_u r'''(t_0) + \Delta'(t_0) + o_p(1). \end{aligned}$$

Here $o_p(1)$ is a r.v. that tends to zero in probability as $u \rightarrow \infty$. This implies that $u(\tau_u - t_0)^{2k_0-1}$ has the same asymptotic distribution as

$$\frac{(2k_0-1)!}{r^{(2k_0)}(t_0)} (\eta_u r'''(t_0) - \Delta'(t_0)).$$

A similar analysis of $\delta_u = u - \xi_u(\tau_u)$ gives the following theorem (Theorem 3.1 in [D]):

The normalized wave-length and amplitude $u(\tau_u - t_0)^{2k_0-1}$ and $\delta_u - u(1-r(t_0))$ are asymptotically independent and normal with mean zero and variances $\{(2k_0-1)!/r^{(2k_0)}(t_0)\}^2 (\lambda_2 - r''(t_0)^2/\lambda_2)$ and $1-r^2(t_0)$ respectively.

The possibility that $r''(t_0) = 0$ is not excluded.

We now turn to the more complicated case where r possesses a sequence of terrace points $t_1 < \dots < t_n$ ($< t_0$) before its first local minimum. Let the first non-vanishing derivatives be $r^{(2k_i+1)}(t_i) < 0$, $i = 1, \dots, n$. Note that $\xi_u(t)$ may have a minimum near any one of the times $t_1, \dots, t_n$. It certainly has one near t_0 , but it needs not be the first one. The asymptotic normal distribution of τ_u has to be modified. The complete result is stated in Theorem 4.1 in [D], which we summarize.

Let $\psi_1, \dots, \psi_n, \psi_0$, and $\chi_1, \dots, \chi_n, \chi_0$ be normally distributed r.v.'s with mean zero and covariances

$$\text{Cov}(\psi_i, \psi_j) = r(t_i - t_j) - r(t_i)r(t_j)$$

$$\text{Cov}(\psi_i, \chi_j) = -r'(t_i - t_j)$$

$$\text{Cov}(\chi_i, \chi_j) = -r''(t_i - t_j) - r''(t_i)r''(t_j)/\lambda_2.$$

If r has terrace points at $t_1 < \dots < t_n$ and a first local minimum at $t_0 > t_n$, then, as $u \rightarrow \infty$,

$$P(\tau_u \text{ is near } t_j) \rightarrow P(\chi_j < 0, \chi_i > 0, i = 1, \dots, j-1),$$

$$j = 1, \dots, n$$

$$P(\tau_u \text{ is near } t_0) \rightarrow P(\chi_i > 0, i = 1, \dots, n).$$

Conditioned on the event that τ_u is near t_j , the asymptotic distributions of $u(\tau_u - t_j)^{2kj}$ (of $u(\tau_u - t_0)^{2k_0-1}$ if $j = 0$) and $\delta_u - u(1-r(t_j))$ are given by the conditional distributions of

$$\frac{(2k_j)!}{n^{(2k_j+1)}(t_j)} \chi_j, \psi_j \mid \chi_j < 0, \chi_i > 0, i = 1, \dots, j-1 \\ (j = 1, \dots, n)$$

and

$$\frac{(2k_0-1)!}{n^{(2k_0)}(t_0)} \chi_0, \psi_0 \mid \chi_i > 0, i = 1, \dots, n \quad (j = 0).$$

b) Suppose that $r'(t) < 0$ for $t > 0$. Then $P(\tau_u < t) \rightarrow 0$ for any $t > 0$, and τ_u will tend to infinity as $u \rightarrow \infty$. The question now is: how fast?

Assume that $r(t) \rightarrow 0$ as $t \rightarrow \infty$, (otherwise ξ is not ergodic). Then all its derivatives will tend to zero as well, and the term $\eta_u(\lambda_2 r(t) + r''(t))$ in $\xi_u(t)$ will be negligible as $u \rightarrow \infty, t \rightarrow \infty$. Furthermore, the covariance function $C(s, t)$ of the non-stationary process $\Delta(t)$ will tend to $r(s-t)$ as $s, t \rightarrow \infty$, which is nothing but the covariance function of the original, unconditioned process $\xi(t)$. This suggests that, for large u and t , the process $\xi_u(t)$ behaves like the process $ur(t) + \xi(t)$.

The main object of Sections 5 and 6 in [D] is to compare the variables

$$v_t = \text{the number of up-crossing zeros of } \xi_u'(s) \text{ in } (0, t]$$

and

$$\mu_t = \text{the number of up-crossing zeros of } ur'(s) + \xi'(s) \text{ in } (0, t].$$

In summarizing the main results, it is convenient to separate the discussion according to the asymptotic behaviour of $r'(t)$. Suppose that $-r''(t)/r'(t)$ has a limit $C, 0 \leq C \leq \infty$, as $t \rightarrow \infty$. This assumption holds for many practically important covariance functions. Actually, we have to impose some further restrictions upon r , the most important of which is condition (6) below, and the condition $r'(t) = O(t^{-\gamma})$ as $t \rightarrow \infty, \gamma > 1$.

Now define a function $T = T(u)$, such that $T \rightarrow \infty$ as $u \rightarrow \infty$, and $E(\mu_T) \rightarrow \theta$, say. According to the value of C , we will write this function as T^0, T^C , or T^∞ .

$C = 0$. If $-r''(t)/r'(t) \rightarrow 0$ as $t \rightarrow \infty$, then $ur'(t)$, which is the dominant term in $\xi_u'(t)$, behaves like a constant for large t -values. This implies that $E(\mu_T)$ (= the expected number of up-crossing zeros of $ur'(t) + \xi'(t)$ in $(0, T]$) can have a finite limit θ only if $ur'(T)$ tends to $-\infty$. We are therefore dealing with crossings of "high" functions by the processes $\Delta'(t)$ and $\xi'(t)$. As is well known, the stream of

crossings of a high level by a stationary normal process is approximately a Poisson process. The main difference here from this classical situation is that we are dealing with non-stationary processes. This complicates proofs, but the main result remains the same. Theorems 5.1 and 5.2 [D] state:

If $r'(t) < 0$ for $t > 0$, $-r''(t)/r'(t) \rightarrow 0$ as $t \rightarrow \infty$, and $T^0 \rightarrow \infty$ as $u \rightarrow \infty$ so that $E(\nu_{T^0}) \rightarrow \theta > 0$, then, for any x ,

$$E(\nu_{T^0+x}) \rightarrow \theta \text{ and } P(\tau_u > T^0+x) \rightarrow e^{-\theta} \text{ as } u \rightarrow \infty.$$

$0 < C \leq \infty$: Here it is easy to determine the function T so that $\lim E(\nu_T)$ is finite. Let T^C and T^∞ tend to infinity so that

$$\begin{aligned} -ur'(T^C) &= 1 & \text{if } 0 < C < \infty \\ -ur'(T^\infty) &= \sqrt{-r'(T^\infty)/r''(T^\infty)} & \text{if } C = \infty \end{aligned} \quad (3)$$

Then, for any fixed x , $-ur'(T^C+x) \rightarrow e^{-Cx}$ if $0 < C < \infty$, while $-ur'(T^\infty+x) \rightarrow \infty$ ($x < 0$), 0 ($x \geq 0$) if $C = \infty$. Theorems 6.1 and 6.2 in [D] state:

If $r'(t) < 0$ for $t > 0$, $-r''(t)/r'(t) \rightarrow C > 0, \leq \infty$ as $t \rightarrow \infty$, and T^C, T^∞ are defined by (3), then, as $u \rightarrow \infty$,

$$\begin{aligned} E(\nu_{T^C+x}) &\rightarrow E \left\{ \begin{array}{l} \text{the number of up-crossing zeros} \\ \text{of } -e^{-Ct} + \xi'(t) \text{ in } (-\infty, x] \end{array} \right\} & \text{if } 0 < C < \infty, \\ E(\nu_{T^\infty+x}) &\rightarrow \begin{array}{l} 0 \\ \frac{1}{2} + E \left\{ \begin{array}{l} \text{the number of up-crossing} \\ \text{zeros of } \xi'(t) \text{ in } (0, x] \end{array} \right\} \end{array} & \begin{array}{l} (x < 0) \\ (x \geq 0) \end{array} \text{ if } C = \infty, \\ P(\tau_u > T^C+x) &\rightarrow P(\sup_{t \leq x} -e^{-Ct} + \xi'(t) < 0) & \text{if } 0 < C < \infty, \\ P(\tau_u > T^\infty+x) &\rightarrow \begin{array}{l} 1 \\ P(\sup_{0 \leq t \leq x} \xi'(t) < 0) \end{array} & \begin{array}{l} (x < 0) \\ (x \geq 0) \end{array} \text{ if } C = \infty. \end{aligned}$$

REMARK. If $r'(t) < 0$ for $t > 0$, then it is not possible to state such an explicit theorem about the distribution of the amplitude δ_u as in the case when r has a local minimum. One general result, however, is Theorem 7.1 in [D]:

If $r'(t) < 0$ for $t > 0$, then $\delta_u/u \rightarrow 1$ in probability as $u \rightarrow \infty$.

If $\lim -r''(t)/r'(t) = C > 0$, then we can say even more. Let the r.v. τ be defined by

$$\tau = \begin{cases} \inf\{t; \xi'(t) \geq e^{-Ct}\} & \text{if } 0 < C < \infty \\ \inf\{t \geq 0; \xi'(t) > 0\} & \text{if } C = \infty. \end{cases}$$

Then $|\tau| < \infty$ (a.s.), and we have, from Theorem 7.2 in [D],

If $r'(t) < 0$ for $t > 0$, and $-r''(t)/r'(t) \rightarrow C > 0$ as $t \rightarrow \infty$, then $\delta_u - u$ is asymptotically (as $u \rightarrow \infty$) distributed as $-C^{-1} \cdot e^{-C\tau} - \xi(\tau)$ if $0 < C < \infty$ and as $-\xi(\tau)$ if $C = \infty$.

A high maximum affects the sample paths of ξ over a wide region around the maximum. A low maximum of course also exerts a strong influence over ξ far beyond the maximum -- as will all extreme $\xi(t)$ -values. But a low maximum exhibits an interesting quality of its own: the sample path will turn upward near the maximum in a rather regular way, depending on the behaviour of $r(t)$ for t near 0.

It is convenient to write the conditioned process $\xi_u(t)$ in a form that differs slightly from (2):

$$\xi_u(t) = uA(t) - \zeta_u B(t) + \Delta(t), \quad (4)$$

where $\zeta_u = (\lambda_4 - \lambda_2^2)\eta_u + \lambda_2 u$, and the functions A and B are defined by

$$A(t) = (\lambda_4 r(t) + \lambda_2 r''(t)) / (\lambda_4 - \lambda_2^2)$$

$$B(t) = (\lambda_2 r(t) + r''(t)) / (\lambda_4 - \lambda_2^2).$$

Then:

The r.v. $|u|\zeta_u$ is asymptotically Γ -distributed as $u \rightarrow -\infty$, with the density

$$\{(\lambda_4 - \lambda_2^2)/\lambda_2\}^{-2} z \exp\{-\lambda_2 z / (\lambda_4 - \lambda_2^2)\}, \quad z > 0.$$

Paper [A] deals with a normalized process

$$\xi^*(t) = |u|^a \{\xi_u(t/|u|^b) - u\}$$

as $u \rightarrow -\infty$. The choice of the normalizing constants a and b , as well as the asymptotic behaviour of ξ^* , will depend on the fine structure of r . We identify two cases.

A) $r^{IV}(t) = \lambda_4 - \lambda_6 t^2/2 + o(t^2)$ as $t \rightarrow 0$. In Section 4 in [A] it appears that the only choice of constants a and b , that leads to non-trivial limit distributions, is $a = 3$, $b = 1$. We cite the results of Theorem 4: Let ω and ζ be two independent r.v.'s, ω a standard normal variable

and ζ a Γ -variable with the density $z \exp(-z)$, $z > 0$. Then:

If r satisfies condition A , the normalized process

$$\xi_A^*(t) = |u|^3 \{ \xi_u(t/|u|) - u \}$$

tends in distribution to the stochastic polynomial

$$\Gamma_A(t) = \frac{\lambda_2 \lambda_6 - \lambda_4^2}{\lambda_4 - \lambda_2^2} \cdot \frac{t^4}{4!} + \omega \sqrt{\frac{\lambda_2 \lambda_6 - \lambda_4^2}{\lambda_2}} \cdot \frac{t^3}{3!} - \zeta \cdot \frac{\lambda_4 - \lambda_2^2}{\lambda_2} \cdot \frac{t^2}{2}$$

as $u \rightarrow -\infty$.

The convergence of ξ_A^* towards Γ_A is discussed in detail in the paper [B], under the additional assumption* that

$$-r^{VI}(t) = \lambda_6 - O(|\log|t||^{-a}) \quad \text{as } t \rightarrow 0 \quad \text{for some } a > 1. \quad (5)$$

If r satisfies (5), then Δ has three times continuously differentiable sample paths with $\Delta(0) = \Delta'(0) = \Delta''(0) = 0$ (a.s.), and

$$|u|^3 \Delta(t/|u|) = \Delta'''(0) \frac{t^3}{3!} + t^3 h(t, u),$$

where $h(t, u) \rightarrow 0$ uniformly in $|t| \leq T$, as $u \rightarrow -\infty$. It is seen that the normal variable $\omega[(\lambda_2 \lambda_6 - \lambda_4^2)/\lambda_2]^{1/2}$, that occurs in the polynomial Γ_A , actually plays the role of $\Delta'''(0)$.

Paper [B] also deals with the wave-length τ_u and the amplitude δ_u . Theorem 3.2 states:

If r satisfies (5), then the normalized wave-length and amplitude

$|u|\tau_u$ and $|u|^3\delta_u$ are asymptotically (as $u \rightarrow -\infty$) distributed as the wave-length and amplitude τ and δ in Γ_A , where

τ = the first (and only) positive minimum of $\Gamma_A(t)$

$$\delta = -\Gamma_A(\tau).$$

* The condition actually used in [B] is: $-r^{VI}(t) = \lambda_6 - O(t)$, which is unnecessarily strong. For consistency with (1) we use the formulation (5) in this summary.

B) $r^{IV}(t) = \lambda_4 - \lambda_5|t| + o(t)$ as $t \rightarrow 0$. Now ξ behaves more irregularly than in case A, and we have to take $a = 5$, $b = 2$.

To re-formulate the results of Theorem 5 in [A], let ζ as before be a Γ -variable with the density $z \exp(-z)$, and let $\omega(t)$ be a Gaussian process, independent of ζ , the second derivative of which is a standard Wiener process $\varepsilon(t)$:

$$\omega(t) = \int_0^t \left\{ \int_0^s \varepsilon(x) dx \right\} ds.$$

If r satisfies condition B, then the normalized process

$$\xi_B^*(t) = |u|^5 \{ \xi_u(t/u^2) - u \}$$

tends in distribution to the process

$$\Gamma_B(t) = \frac{\lambda_2 \lambda_5}{\lambda_4 - \lambda_2^2} \cdot \frac{|t|^3}{3!} + \sqrt{2\lambda_5} \omega(t) - \zeta \cdot \frac{\lambda_4 - \lambda_2^2}{\lambda_2} \cdot \frac{t^2}{2}$$

as $u \rightarrow \infty$.

The exact distribution of the wave-length and amplitude for finite u -values is not easily obtained. Even for the more simple problem of the time distance between one maximum and the following minimum, regardless of their heights, exact solutions are known only for a few covariance functions.

From a practical point of view, however, good approximative solutions can be just as useful. Paper [C] deals with this problem.

WAVE-LENGTH τ_u : Let v_t denote the number of local minima of $\xi_u(s)$ in $(0, t]$. Then

$$E(v_t) - \frac{1}{2} E(v_t(v_t-1)) \leq P(\tau_u \leq t) = P(v_t \geq 1) \leq E(v_t).$$

The expectations can be computed, and functions $f_{(1)}$ and $f_{(2)}$ defined, such that

$$E(v_t) = \int_0^t f_{(1)}(s) ds, \quad E(v_t) - \frac{1}{2} E(v_t(v_t-1)) = \int_0^t f_{(2)}(s) ds.$$

The exact form of $f_{(1)}$ and $f_{(2)}$ is given in [C], Section 1.3. The condition for the integrability of $f_{(1)}$ given in [C] is unnecessarily strong. A weaker, sufficient condition for $f_{(1)}$ to be integrable is condition C3.c in [D], Section 5, which includes the requirement that

$$\int_0^1 s^{-1} \sqrt{\lambda_u - r^{IV}(s)} ds < \infty, \quad (6)$$

If r satisfies certain regularity conditions, then the distribution of the wave-length τ_u is absolutely continuous, and its density function f_1 obeys the inequalities

$$f_{(2)}(t) \leq f_1(t) \leq f_{(1)}(t). \quad (7)$$

The usefulness of (7) will of course depend upon the covariance function r . In [C], Section 1.4, some numerical illustrations are given for four different such functions. Here we will discuss two of them:

$$r_1(t) = \frac{\sin\sqrt{3}t}{\sqrt{3}t}, \quad r_3(t) = e^{-t^2/2}.$$

See Figures 1.1 and 1.3a, b. The number I_2 appearing in the figures denotes the integral of $f_{(2)}(t)$ over the region in which it is non-negative. Since $f_{(2)}(t)$ is a lower bound of $f_1(t)$, a high value of I_2 will indicate that $f_{(2)}(t)$ is a good approximation of $f_1(t)$. For both r_1 and r_3 , the integral I_2 is near one.

One major difference between r_1 and r_3 is that for r_1 the functions $f_{(1)}$ and $f_{(2)}$ are close together as long as $f_{(2)}(t)$ is positive, while for r_3 , the function $f_{(1)}$ deviates from $f_{(2)}$ much earlier. This difference is of importance in the discussion of the amplitude distribution.

AMPLITUDE δ_u : Not much is known about its distribution. Since $\delta_u = u - \xi_u(\tau_u)$, we have to evaluate the process at a random time, depending on $\xi_u(t)$ for $0 \leq t \leq \tau_u$. One approach, that proposes itself, is to condition the process upon the value of τ_u , i.e. to look at

$$\xi_u(\tau) \mid \xi_u(t) \text{ has its first local minimum at } \tau. \quad (8)$$

The condition in (8) is, however, too complicated to handle, and one could like to replace it by a simpler one; for instance

$$\xi_u(\tau) \mid \xi_u(t) \text{ has a local minimum at } \tau. \quad (9)$$

Will the distribution obtained in this way differ considerably from the true amplitude distribution? For many processes, it does not.

Suppose there is a time $t_0 > 0$ such that, with a probability close to one, $\xi_u(t)$ has one and only one local minimum in $(0, t_0]$. For such a process, the two conditions in (8) and (9) are approximately equivalent, at least for $0 \leq \tau \leq t_0$.

Therefore, let $\{\xi_{u,\tau}(t), t \in \mathbb{R}\}$ be the doubly-conditioned process that has a u -maximum at 0 and a local minimum at τ . Theorem 2.2 in [C] states:

The process $\xi_{u,\tau}(t)$ has the same distributions as a process

$$uA_\tau(t) + \Delta_\tau(t) - \zeta_{\tau 1}B_{\tau 1}(t) + \zeta_{\tau 2}B_{\tau 2}(t),$$

where A_τ , $B_{\tau 1}$, and $B_{\tau 2}$ are certain functions, $\zeta_{\tau 1}$ and $\zeta_{\tau 2}$

are r.v.'s, and Δ_τ is a non-stationary Gaussian process.

Cf. the representation (4) of $\xi_u(t)$.

Now let $f_{\xi_{u,\tau}}(\tau)$ be the density function of the r.v. $\xi_{u,\tau}(\tau)$, i.e. of the value of the process $\xi_{u,\tau}$ at the minimum point, and define

$$f_{(\tau_u, \delta_u)}(t, x) = f_{(1)}(t) \cdot f_{\xi_{u,t}}(t)(u-x)$$

$$f_{(\delta_u)}(x) = \int_0^{t_0} f_{(\tau_u, \delta_u)}(t, x) dt.$$

Theorem 2.4 in [C] states:

If there is a time $t_0 > 0$ such that, with a probability near one, $\xi_u(t)$ has one and only one local minimum in $(0, t_0]$, then the wave-length τ_u and amplitude δ_u are jointly absolutely continuous over $(0, t_0] \times (0, \infty)$, and their density function f_{τ_u, δ_u} can be approximated by $f_{(\tau_u, \delta_u)}$ in that region.

Under the same condition, the amplitude distribution can be approximated by $f_{(\delta_u)}$.

The error in the approximations can be estimated in both cases. Some numerical illustrations of $f_{(\tau_u, \delta_u)}$ and $f_{(\delta_u)}$ are given in [C], Section 2.3, for two different covariance functions.

Let $\{\xi(\underline{t}), \underline{t} \in \mathbb{R}^n\}$ be a separable, homogeneous Gaussian field with n -dimensional time parameter, mean zero and covariance function r . If $r(\underline{t})$ depends only on the length $|\underline{t}|$, i.e. if there is a function r_* , ($R \sim R$), such that

$$r(\underline{t}) = r_*(|\underline{t}|),$$

then the field is isotropic. Under suitable regularity conditions upon r and its four first partial derivatives, the sample paths of ξ are twice continuously differentiable, with probability one.

Paper [E] generalizes most of the results obtained in [A], [B], and [D] for the case with n -dimensional time.

Consider the event " ξ has a local minimum with height in $(u, u+h)$ for some $\underline{g}$ in the sphere $|\underline{g}| \leq h'$ ". Compute the conditional distributions of $\xi(\underline{t})$ given that this event has occurred, and let $h, h' \neq 0$. Name the resulting distributions "*the conditional distributions of $\xi(\underline{t})$, given that ξ has local u -maximum at $\underline{0}$* ". Let $\{\xi_u(\underline{t}), \underline{t} \in \mathbb{R}^n\}$ be a random field that has these distributions. Then $\xi_u(\underline{t})$ can be represented as

$$\xi_u(\underline{t}) = u r(\underline{t}) - \underline{\eta}_u \underline{b}(\underline{t}) + \Delta(\underline{t}) \quad (10)$$

$$= u A(\underline{t}) - \underline{\zeta}_u \underline{b}(\underline{t}) + \Delta(\underline{t}), \quad (11)$$

where A is a function $\mathbb{R}^n \sim \mathbb{R}$, $\underline{b}$ is a vector function $\mathbb{R}^n \sim \mathbb{R}^{n(n+1)/2}$, while $\underline{\eta}_u$ and $\underline{\zeta}_u$ are $n(n+1)/2$ -dimensional random vectors with certain distributions, and Δ is a non-homogeneous Gaussian field, independent of $\underline{\eta}_u$ and $\underline{\zeta}_u$.

HIGH MAXIMA: Suppose r has a strict local minimum at $\underline{t}^0$, and that the matrix $R_{\underline{t}^0}$ of second order derivatives of r is positive definite. This can happen only in the non-isotropic case. Then the process $\xi_u(\underline{t})$, expressed as (10), will have a local minimum near $\underline{t}^0$, say at $\underline{\tau}^u$. Theorem 3.1 in [E] states:

The r.v.'s $u(\underline{\tau}^u - \underline{t}^0)$ and $\xi_u(\underline{\tau}^u) - ur(\underline{t}^0)$ are asymptotically independent and n -variate and uni-variate normal respectively, as $u \rightarrow \infty$.

If $R_{\underline{t}^0}$ is not positive definite, then $u(\underline{\tau}^u - \underline{t}^0)$ is not normal. Under certain conditions, however, the asymptotic normality holds after a transformation of the region near $\underline{t}^0$. Theorem 3.2 in [E] gives details.

In the isotropic case, $r(\underline{t}) = r_*(|\underline{t}|)$, we can obtain results which have no direct counterpart in the one-dimensional case. Since $r(\underline{t})$ can have no strict local minimum, the locations of the (strict) local minima of $\xi_u(\underline{t})$ are less precisely determined than in the non-isotropic case. Suppose r_* has a first (strict) local minimum at $t_0 > 0$. Then r has a (non-strict) local minimum for any $\underline{t}$ with $|\underline{t}| = t_0$, and $\xi_u(\underline{t})$ will tend to have (strict) local minima near the surface $|\underline{t}| = t_0$.

Thus it seems natural to look at $\xi_u(\underline{t})$ along fixed directions $\underline{\theta}$ from the origin. Let

$$\xi_u^t(\underline{\theta}) = \xi_u(t \cdot \underline{\theta}), \quad (t \geq 0),$$

(where $\underline{\theta} = (\theta_1, \dots, \theta_n)$, $|\underline{\theta}| = 1$, defines a direction), be the ξ_u -process observed along $\underline{\theta}$. The local minima of $\xi_u(\underline{t})$ are then to be found among those $\underline{t}$ for which $\xi_u^s(\underline{t}/|\underline{t}|)$, regarded as a function of s , has a local minimum for $s = |\underline{t}|$.

For each $\underline{\theta}$, let $\tau_u(\underline{\theta})$ be the t -value for which $\xi_u^t(\underline{\theta})$ attains its first local minimum. Then $\tau_u(\underline{\theta})$ and $\xi_u(\tau_u(\underline{\theta}) \cdot \underline{\theta})$ define two stochastic fields over the unit sphere $|\underline{\theta}| = 1$. Theorem 3.3 in [E] states:

If ξ is isotropic, $r(\underline{t}) = r_*(|\underline{t}|)$, and r_* has a first local minimum at t_0 with first non-vanishing derivative $r_*^{(2k)}(t_0) > 0$, then the stochastic fields

$$u(\tau_u(\underline{\theta}) - t_0)^{2k-1}, \quad \xi_u(\tau_u(\underline{\theta}) \cdot \underline{\theta}) - ur_*(t_0); \quad |\underline{\theta}| = 1$$

tend in distribution to two dependent Gaussian fields,

$$\frac{(2k-1)!}{r_*^{(2k)}(t_0)} \sigma^0(\underline{\theta}), \quad -\psi^0(\underline{\theta}); \quad |\underline{\theta}| = 1,$$

where

$$V(\sigma^O(\underline{\theta})) = \lambda_2^* - r_*''(t_0)^2 / \lambda_2^*$$

$$V(\psi^O(\underline{\theta})) = 1 - r_*^2(t_0)$$

$$\text{Cov}(\sigma^O(\underline{\theta}), \psi^O(\underline{\theta})) = 0$$

$$\lambda_2^* = - r_*''(t_0).$$

LOW MAXIMA: Normalize $\xi_u(\underline{t})$, using (11):

$$\xi_u^*(\underline{t}) = |u|^3 \{ \xi_u(\underline{t}/|u|) - u \}.$$

Theorem 2.1 in [E] states:

If r is six times continuously differentiable and satisfies certain other regularity conditions, then $\xi_u^(\underline{t})$ tends in distribution to a fourth degree polynomial in $t_1, \dots, t_n$ with random coefficients for the third and second order terms.*

The proof given in [E] works only for isotropic processes, but the arguments indicate that the theorem is true in the general case.

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